

DETECTING FIXED POINTS OF NONEXPANSIVE MAPS
BY ILLUMINATING THE UNIT BALL*

BY

BAS LEMMENS

*School of Mathematics, Statistics, and Actuarial Science, University of Kent,
Canterbury, Kent CT2 7NZ, England
e-mail: B.Lemmens@kent.ac.uk*

AND

BRIAN LINS**

*Department of Mathematics and Computer Science, Hampden-Sydney College,
Hampden-Sydney, VA 23943, USA
e-mail: blins@hsc.edu*

AND

ROGER NUSSBAUM

*Department of Mathematics, Rutgers University,
Hill Center for the Mathematical Sciences,
110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA
e-mail: nussbaum@math.rutgers.edu*

* This work was partially supported by NSF grant DMS-1201328.

** Corresponding author.

Received July 5, 2016 and in revised form January 14, 2017

ABSTRACT

We give necessary and sufficient conditions for a nonexpansive map on a finite-dimensional normed space to have a nonempty, bounded set of fixed points. Among other results we show that if $f : V \rightarrow V$ is a nonexpansive map on a finite-dimensional normed space V , then the fixed point set of f is nonempty and bounded if and only if there exist w_1, \dots, w_m in V such that $\{f(w_i) - w_i : i = 1, \dots, m\}$ illuminates the unit ball. This yields a numerical procedure for detecting fixed points of nonexpansive maps on finite-dimensional spaces. We also discuss applications of this procedure to certain nonlinear eigenvalue problems arising in game theory and mathematical biology.

1. Introduction

A central problem in metric fixed point theory is to understand when a nonexpansive map $f : X \rightarrow X$ on a metric space (X, d) has a fixed point. There are numerous results when X is a closed, bounded, convex subset in a Banach space V ; see [16, 17]. Of course, if V is finite-dimensional and $X \subset V$ is compact and convex, then the Brouwer fixed point theorem immediately resolves the question. If, however, X is unbounded, it is not at all clear when f has a fixed point, even if the normed space V is finite-dimensional.

In this paper we study the fixed point set, $\text{Fix}(f)$, of nonexpansive maps on finite-dimensional normed spaces. For such maps, there are many algorithms known to approximate fixed points [21, 22, 26, 34], if one exists. The results of this paper complement these algorithms by providing computational methods that can confirm the existence of fixed points. The paper is organized as follows.

In Section 2 below we review the horofunction compactification of a complete, proper, metric space. Our Proposition 2.2 extends a result of Beardon [5, Proposition 4.5] and allows us to give necessary and sufficient conditions for the fixed point set of a nonexpansive map to be bounded and nonempty.

In Section 3 we focus on the fixed point set of maps $f : V \rightarrow V$ when V is a finite-dimensional normed space. Our main result, Theorem 3.4, gives necessary and sufficient conditions for $\text{Fix}(f)$ to be nonempty and bounded. In particular, we show that $\text{Fix}(f)$ is nonempty and bounded if and only if there exist a finite number of points w_1, \dots, w_m in V such that

$$S := \{f(w_i) - w_i : i = 1, \dots, m\}$$

illuminates the closed unit ball B_1 of V . Recall that S illuminates B_1 if for each $w \in \partial B_1$ there exists $s \in S$ such that $w + \lambda s \in \text{int } B_1$ for all $\lambda > 0$ sufficiently small. Interestingly, it is a famous unresolved conjecture whether every compact, convex body in an n -dimensional vector space V can be illuminated by some subset S of V with cardinality less than or equal to 2^n .

Theorem 3.4 suggests the following simple procedure for detecting fixed points of nonexpansive maps $f : V \rightarrow V$. Generate randomly a finite set S in V and check if $\{f(w) - w : w \in S\}$ illuminates the unit ball of V . In Section 4 we discuss criteria that can be verified computationally to check if a set illuminates the unit ball for a variety of norms. In Section 5, we apply the results to certain nonlinear eigenvalue problems that arise in game theory [7, 38] and mathematical biology [29, 39], and perform some numerical experiments to test the feasibility of the procedure. Of particular interest is Theorem 5.1 which is a nonlinear Perron–Frobenius theorem. The final section explains how illuminating sets can be used to place bounds on the location of the fixed point set of nonexpansive maps.

2. Fixed points and horofunctions

Throughout the paper we will use the notation $\text{cl } A$, $\text{int } A$ and ∂A to, respectively, denote the closure, interior and boundary of a set A in a topological space. We will also denote the closed ball with radius $r > 0$ and center x by $B_r(x)$.

Let us briefly recall the horofunction compactification of a complete, proper, metric space (X, d) ; see [10, 19, 36]. Here, *proper* means that every closed ball of radius $R \geq 0$ in X is compact. Let $C(X)$ denote the space of continuous functions $f : X \rightarrow \mathbb{R}$ equipped with the topology of compact convergence; see [27, §46]. Fix a *base point* $b \in X$ and define for $x \in X$ the function $\tau_b(x) : X \rightarrow \mathbb{R}$ by

$$\tau_b(x)(y) := d(y, x) - d(b, x) \quad \text{for all } y \in X.$$

It is easy to check that for each $x \in X$ the function $\tau_b(x)$ is Lipschitz with constant 1, and hence $\tau_b(X) \subseteq C(X)$ is an equicontinuous family. Moreover, for each fixed $y \in X$ we have that $\{\tau_b(x)(y) : x \in X\} \subseteq [-d(y, b), d(y, b)]$. Thus, it follows from Ascoli’s theorem [27, Theorem 47.1] that $\tau_b(X)$ has compact closure in $C(X)$. The *horofunction boundary* of (X, d) is given by $X(\infty) := \text{cl } \tau_b(X) \setminus \tau_b(X)$, and its elements are called *horofunctions*. Given $h \in X(\infty)$ the set $H(h, r) := \{x \in X : h(x) \leq r\}$ is called the *horoball* with center h and radius $r \in \mathbb{R}$. As (X, d) is proper, it is σ -compact, i.e., X is the union

of countably many open sets with compact closure, and hence the topology of compact convergence is metrizable; see [27, Exercise 10, p. 289]. This implies that every sequence $\tau_b(x_n)$ in $C(X)$ has a convergent subsequence.

If we furthermore assume that the complete, proper, metric space (X, d) is *geodesic*, that is to say, for each $x \neq y$ in X there exists a path $\gamma: [\alpha, \beta] \rightarrow (X, d)$ such that $\gamma(\alpha) = x$, $\gamma(\beta) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $\alpha \leq s \leq t \leq \beta$, then the horofunctions are precisely the limits of converging sequences $\tau_b(x_n)$ such that $d(b, x_n) \rightarrow \infty$. Indeed the following lemma holds, which is a slightly weaker result than [36, Theorem 4.7] by Rieffel, who showed that the horofunctions are precisely the limits of so-called weakly geodesic rays. For completeness we include a proof.

LEMMA 2.1: *If (X, d) is a complete, proper, geodesic metric space, then $h \in X(\infty)$ if and only if there exists a sequence (x_n) in X with $d(b, x_n) \rightarrow \infty$ such that $\tau_b(x_n)$ converges to $h \in \text{cl } \tau_b(X)$ as $n \rightarrow \infty$.*

Proof. If $h \in X(\infty)$, then there exists a sequence (x_n) in X such that $\tau_b(x_n) \rightarrow h$, since the topology of compact convergence is metrizable whenever (X, d) is proper. Note that $d(b, x_n) \rightarrow \infty$. Indeed, otherwise (x_n) has a bounded subsequence (x_{n_k}) which converges to some point say $x \in X$, as (X, d) is a proper metric space. This implies that $h(y) = \lim_{k \rightarrow \infty} \tau_b(x_{n_k})(y) = \tau_b(x)(y)$ for all $y \in X$, and hence h is not a horofunction, which is a contradiction.

To prove sufficiency, we argue by contradiction. Suppose that $h \notin X(\infty)$. Then there exists $x_0 \in X$ such that $h(y) = \tau_b(x_0)(y)$ for all $y \in X$. Let $r := d(b, x_0) + 1$ and $0 < \varepsilon < 1$. As $\tau_b(x_n) \rightarrow h$, there exists $N \geq 1$ such that

$$\sup_{y \in B_r(x_0)} |\tau_b(x_n)(y) - h(y)| < \varepsilon \quad \text{and} \quad d(x_0, x_n) \geq r \quad \text{for all } n \geq N,$$

as the closed ball $B_r(x_0)$ is compact and $d(b, x_n) \rightarrow \infty$ as $n \rightarrow \infty$. Fix $n \geq N$ and let $\gamma_n: [0, \beta_n] \rightarrow X$ be a geodesic from x_0 to x_n . Put $z := \gamma_n(r)$, so $d(x_0, z) = r$, and note that

$$\begin{aligned} |\tau_b(x_0)(z) - h(z)| &\geq |\tau_b(x_0)(z) - \tau_b(x_n)(z)| - |\tau_b(x_n)(z) - h(z)| \\ &\geq |d(z, x_0) - d(b, x_0) - d(z, x_n) + d(b, x_n)| - \varepsilon \\ &= |1 - d(z, x_n) + d(b, x_n)| - \varepsilon \\ &\geq 1 - d(z, x_n) + d(b, x_n) - \varepsilon. \end{aligned}$$

However,

$$d(b, x_n) \geq -d(b, x_0) + d(x_0, x_n) = -d(b, x_0) + d(x_0, z) + d(z, x_n) = 1 + d(z, x_n).$$

Substituting this lower estimate for $d(b, x_n)$ in the expression above gives a lower estimate for $|\tau_b(x_0)(z) - h(z)|$ of $2 - \varepsilon > 1$, which is a contradiction. ■

Recall that $f: X \rightarrow X$ is a *nonexpansive* map on a metric space (X, d) if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. For complete, proper, geodesic metric space (X, d) we introduce the following two properties:

- A1. For each nonexpansive map $f: X \rightarrow X$ there exists a sequence of nonexpansive maps $f_n: X \rightarrow X$ such that each f_n has a fixed point in X and $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for all $x \in X$.
- A2. If $f: X \rightarrow X$ is a nonexpansive map and there exists a closed ball B in (X, d) and a horoball $H(h, r)$ in X such that $f(B \cap H(h, r)) \subseteq B \cap H(h, r)$, then f has a fixed point in $B \cap H(h, r)$.

All finite-dimensional normed spaces satisfy (A1) and (A2). Indeed, their horoballs are closed convex sets [41], so that (A2) follows from the Brouwer fixed point theorem. Furthermore, the maps $f_n := (1 - \frac{1}{n})f$ are Lipschitz contractions, which have unique fixed points, and hence (A1) holds. Other interesting metric spaces that satisfy the properties include Hilbert's metric spaces [25, 28, 33, 42], Thompson's metric on finite-dimensional cones [25, 28, 40], and hyperbolic spaces [10]. Also note that a Busemann metric space satisfies property (A1). Indeed, if (X, d) is Busemann, then we can fix $x_0 \in X$ and define for each $x \in X$ an affinely re-parametrized geodesic $\gamma_x: [0, 1] \rightarrow X$ connecting x_0 to x , so $d(\gamma_x(s), \gamma_x(t)) = |t - s|d(x, x_0)$ for all $s, t \in [0, 1]$. The geodesics γ_x satisfy

$$(2.1) \quad d(\gamma_x(s), \gamma_y(s)) \leq sd(x, y) \quad \text{for all } x, y \in X \text{ and } s \in [0, 1],$$

as (X, d) is Busemann; see [32, Proposition 8.1.2]. It follows that the maps $r_s: X \rightarrow X$ given by $r_s(x) = \gamma_x(s)$ are Lipschitz contractions. Thus, for each $n \geq 1$ the map $f_n: X \rightarrow X$ given by $f_n(x) = f(r_{1-1/n}(x))$ is a Lipschitz contraction on X that satisfies

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(r_{1-1/n}(x), x) \\ &= d(\gamma_x(1 - 1/n), \gamma_x(1)) \\ &= \frac{1}{n}d(x, x_0) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, for all $x \in X$, and hence property (A1) holds. In fact, one does not need all geodesics to satisfy the Busemann convexity property (2.1) as long as there are enough to define the Lipschitz contractions r_s . Such metric spaces have been studied in [15] and are called metrically star-shaped.

The following proposition extends a result by Beardon [5, Proposition 4.5]. The reader should note that Beardon makes some additional assumptions that are not required in our setting.

PROPOSITION 2.2: *If $f: X \rightarrow X$ is a nonexpansive map on a complete, proper, geodesic metric space (X, d) satisfying properties (A1) and (A2), then $\text{Fix}(f)$ is empty or unbounded if and only if there exists $h \in X(\infty)$ such that $h(f(x)) \leq h(x)$ for all $x \in X$.*

Proof. The assertion that if $\text{Fix}(f) = \emptyset$ or unbounded, then there exists $h \in X(\infty)$ such that $h(f(x)) \leq h(x)$ for all $x \in X$, is due to Beardon [5, Proposition 4.5]. For completeness we include the argument. Suppose that $\text{Fix}(f) = \emptyset$. As (X, d) satisfies (A1), there exists a sequence of nonexpansive maps $f_n: X \rightarrow X$ such that each f_n has a fixed point $x_n \in X$ and $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for all $x \in X$. Note that $d(b, x_n) \rightarrow \infty$ as $n \rightarrow \infty$, as otherwise there exists a convergent subsequence (x_{n_k}) , with limit say z , as (X, d) is proper. Clearly

$$d(z, f(z)) \leq d(z, x_{n_k}) + d(f_{n_k}(x_{n_k}), f_{n_k}(z)) + d(f_{n_k}(z), f(z)) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

so that z is a fixed point of f , which is impossible.

By taking a subsequence we may assume that $\tau_b(x_n)$ converges to $h \in X(\infty)$ by Lemma 2.1. Note that for each $x \in X$ we have that

$$\begin{aligned} d(f(x), x_n) - d(b, x_n) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_n)) - d(b, x_n) \\ &\leq d(f(x), f_n(x)) + d(x, x_n) - d(b, x_n). \end{aligned}$$

By taking limits, we deduce that $h(f(x)) \leq h(x)$ for all $x \in X$. If $\text{Fix}(f)$ is unbounded, then we can use an unbounded sequence of fixed points (x_n) of f to create a horofunction h such that $h(f(x)) \leq h(x)$ for all $x \in X$.

To prove the converse statement suppose that there exists a horofunction h with $h(f(x)) \leq h(x)$ for all $x \in X$. By Lemma 2.1 there exists a sequence (z_n) in X such that $\tau_b(z_n) \rightarrow h$ and $d(z_n, b) \rightarrow \infty$. Consider the horoballs $H_{-r} := H(h, -r) = \{x \in X : h(x) \leq -r\}$ for $r \geq 0$. Note that H_{-r} is nonempty. Indeed, for each $n \geq 1$ such that $d(b, z_n) > r$, there exists $y_n \in \partial B_r(b)$ such that $d(z_n, b) = d(z_n, y_n) + d(y_n, b)$, as (X, d) is a geodesic space. As $\partial B_r(b)$ is

compact, there exists a subsequence (y_{n_k}) such that $y_{n_k} \rightarrow y^*$. Thus, for each $\varepsilon > 0$ there exists $K \geq 1$ such that $h(y^*) - \varepsilon < \tau_b(z_{n_k})(y^*)$ and $d(y^*, y_{n_k}) < \varepsilon$ for all $k \geq K$. We have that

$$\begin{aligned} \tau_b(z_{n_k})(y^*) &= d(y^*, z_{n_k}) - d(z_{n_k}, b) \\ &\leq d(y^*, y_{n_k}) + d(y_{n_k}, z_{n_k}) - d(z_{n_k}, b) \\ &< \varepsilon - d(y_{n_k}, b) = \varepsilon - r, \end{aligned}$$

which implies that $h(y^*) \leq -r$ as $\varepsilon > 0$ is arbitrary.

Note that if $y \in H_{-r}$, then for each $\varepsilon > 0$ small and each n sufficiently large we have that

$$d(y, z_n) - d(b, z_n) \leq -r + \varepsilon,$$

so that $d(y, b) \geq d(b, z_n) - d(y, z_n) \geq r - \varepsilon$. Thus, $d(b, y) \geq r$ for all $y \in H_{-r}$.

Now suppose that $\text{Fix}(f)$ is nonempty, and let $z \in \text{Fix}(f)$. For $m \geq 1$ pick $w_m \in H_{-m}$ and put $r_m := d(w_m, z)$. Then we know that

$$f(B_{r_m}(z) \cap H_{-m}) \subseteq B_{r_m}(z) \cap H_{-m}, \quad \text{for all } m \geq 1.$$

By property (A2), f has a fixed point $v_m \in B_{r_m}(z) \cap H_{-m}$ for each $m \geq 1$. But then $d(v_m, b) \geq m$, which shows that $\text{Fix}(f)$ is unbounded, which completes the proof. ■

For a finite-dimensional normed space $(V, \|\cdot\|)$, let $(V^*, \|\cdot\|_*)$ denote the dual space. Recall that a norm is *smooth* if for every $z \in V$ with $\|z\| = 1$, there is a unique $\varphi \in V^*$, $\|\varphi\|_* = 1$, such that $\varphi(z) = 1$. It is a consequence of [36, Proposition 6.2] that for a finite-dimensional vector space with a smooth norm, the horofunctions are given by $h(x) = -\varphi(x - b)$ for $x \in V$, where $b \in V$ and $\varphi \in V^*$ are fixed with $\|\varphi\|_* = 1$. This gives rise to the following generalization of [4, Theorem 7].

COROLLARY 2.3: *If $f: V \rightarrow V$ is a nonexpansive map on a finite-dimensional normed space $(V, \|\cdot\|)$ with a smooth norm, then $\text{Fix}(f)$ is empty or unbounded if and only if there exists a nonzero linear functional $\varphi \in V^*$ such that $\varphi(f(x)) \geq \varphi(x)$ for all $x \in V$.*

REMARK 2.4: Part of Corollary 2.3 does not depend on the smoothness of the norm. Indeed, suppose that C is a closed, convex subset of a finite dimensional normed linear space $(V, \|\cdot\|)$, where the norm is not necessarily smooth. Let $f: C \rightarrow C$ be a nonexpansive map and suppose that φ is a nonzero linear

functional on V such that (i) $\{x \in C: \varphi(x) \geq a\}$ is nonempty for all a , and (ii) $\varphi(f(x)) \geq \varphi(x)$ for all $x \in V$. Then $\text{Fix}(f)$ is empty or unbounded.

To show this, let $H_m := \{x \in C: \varphi(x) \geq m\}$, $m \in \mathbb{N}$, so H_m is closed, convex and nonempty. Assume $\text{Fix}(f)$ is nonempty and take $z \in \text{Fix}(f)$ and $w_m \in H_m$ and define $r_m := \|z - w_m\|$. By our construction $w_m \in B_{r_m}(z) \cap H_m$ and $B_{r_m}(z) \cap H_m$ is closed, bounded and convex. Because f is nonexpansive and condition (ii) above is satisfied, $f(B_{r_m}(z) \cap H_m) \subseteq B_{r_m} \cap H_m$, so Brouwer's fixed point theorem implies that f has a fixed point $x_m \in B_{r_m} \cap H_m$. Since $\varphi(x_m) \geq m$, $\|x_m\| \geq m/\|\varphi\|$ and $\text{Fix}(f)$ is unbounded.

Proposition 2.2 gives a criterion for the existence of a fixed point in terms of horofunctions and one may wonder for which metric spaces this criterion can be verified computationally. In this paper we will see that for finite-dimensional normed spaces there is a nice way to check the criterion by using so-called illuminating vectors.

3. Fixed points in normed spaces

Given a compact convex set K in a finite-dimensional vector space V with nonempty interior, we say that $x \in \partial K$ is *illuminated by* $w \in V$ if $x + \lambda w \in \text{int } K$ for some $\lambda > 0$. A set $S \subseteq V$ is said to *illuminate* K if each point in ∂K is illuminated by some $w \in S$. The *illumination number* of K is defined by

$$c(K) := \min\{|S|: S \subset V \text{ illuminates } K\}.$$

The illumination number was introduced by Boltjanski [8], who showed that it is equal to the so-called *covering number* $b(K)$ of K , which is the smallest number of strictly smaller homothetical copies K_1, \dots, K_m of K which cover K , so $K \subseteq \bigcup_{i=1}^m K_i$. It is clear by compactness that $b(K)$ is a finite number. Gohberg and Markus [18] conjectured that $b(K) \leq 2^n$ for any compact, convex body K in V , where $n = \dim(V)$. Moreover, equality holds if and only if K is an n -dimensional parallelepiped. Hadwiger [20] also independently raised the question of the maximal value of $b(K)$. Gohberg and Markus's conjecture is commonly referred to as the Illumination Conjecture, and remains unsolved for general compact convex sets. A detailed survey is given in [9, Chapter VI].

Before stating our main result, let us recall the definition of the topological degree in finite-dimensional vector spaces V . Given an open, bounded set $G \subseteq V$ and a continuous map $f: \text{cl } G \rightarrow V$ such that $f(x) \neq a$ for all $x \in \partial G$, there

exists an integer $\deg(f, G, a)$ called the *topological degree of f on G with respect to a* , which has the following properties:

- D1. $\deg(\text{id}, G, a) = 1$ if $a \in G$ and $\deg(\text{id}, G, a) = 0$ if $a \notin \text{cl}G$, where id is the identity map on V .
- D2. (Additivity Property) If G_1 and G_2 are disjoint, open subsets of G and if $a \notin f(\text{cl}G \setminus (G_1 \cup G_2))$, then

$$\deg(f, G, a) = \deg(f, G_1, a) + \deg(f, G_2, a).$$

- D3. (Homotopy Property) Let $F: \text{cl}G \times [0, 1] \rightarrow V$ and $a: [0, 1] \rightarrow V$ be continuous maps, and let $f_t: \text{cl}G \rightarrow V$ be given by $f_t(x) := F(x, t)$ for all $t \in [0, 1]$. If $a(t) \notin f_t(\partial G)$ for all $t \in [0, 1]$, then $\deg(f_t, G, a(t))$ is constant for all $t \in [0, 1]$.

We allow the possibility that G , G_1 , or G_2 is empty, so (D2) gives $\deg(f, \emptyset, a) = 0$. Notice this implies that if $\deg(f, G, a) \neq 0$, there exists $x \in \text{cl}G$ with $f(x) = a$, but *not* conversely. Amann and Weiss [2] proved that properties (D1), (D2) and (D3) uniquely determine the topological degree. We refer the reader to [13] for further details.

The following proposition is almost certainly known, but we are unaware of a reference.

PROPOSITION 3.1: *Let G be an open, bounded subset of a finite-dimensional normed space $(V, \|\cdot\|)$, and suppose that $g: \text{cl}G \rightarrow V$ is a continuous map with $g(x) \neq 0$ for all $x \in \partial G$. If $\deg(g, G, 0) \neq 0$, then for each $b \in V$ with $\|b\| = 1$ there exists $x \in \partial G$ such that $b = g(x)/\|g(x)\|$.*

Proof. We argue by contradiction. So, suppose that there exists $b \in V$ with $\|b\| = 1$ such that $g(x) \neq \|g(x)\|b$ for all $x \in \partial G$. Then for each $x \in \partial G$ and each $\lambda > 0$ we have that $g(x) \neq \lambda b$. Now define for $t \in [0, 1]$ a continuous map $g_t: \text{cl}G \rightarrow V$ by

$$g_t(x) = (1 - t)g(x) - tb \quad \text{for all } x \in \text{cl}G.$$

Note that for $0 \leq t < 1$ we have that $g_t(x) \neq 0$ for all $x \in \partial G$, as otherwise $g(x) = \frac{t}{1-t}b$ for some $x \in \partial G$. Also $g_1(x) = -b \neq 0$ for all $x \in \text{cl}G$. So the homotopy property (D3) gives $\deg(g, G, 0) = \deg(g_1, G, 0) = 0$, which is a contradiction. ■

We also have the following proposition.

PROPOSITION 3.2: *Let D be an open subset of a finite dimensional normed space $(V, \|\cdot\|)$. Suppose that $f: D \rightarrow V$ is a nonexpansive map such that $\text{Fix}(f)$ is compact and nonempty. Then $\text{Fix}(f)$ is connected. If G is a bounded open set such that $\text{Fix}(f) \subset G$ and $\text{cl} G \subset D$, then $\text{deg}(\text{id} - f, G, 0) = 1$.*

Proof. By our assumptions $x \neq f(x)$ for all $x \in \partial G$, since $\text{Fix}(f) \subset G$ and G is open. Therefore $\text{deg}(\text{id} - f, G, 0)$ is defined. More generally, suppose that H is a bounded open subset such that $H \cap \text{Fix}(f)$ is nonempty, $\text{cl} H \subset D$, and $f(x) \neq x$ for all $x \in \partial H$. Because ∂H is a compact set and f is a continuous map such that $\|x - f(x)\| > 0$ for all $x \in \partial H$, there exists $c > 0$ with $\|x - f(x)\| \geq c$ for all $x \in \partial H$.

By assumption, there exists $x_0 \in \text{Fix}(f) \cap H$. For $0 \leq t \leq 1$ define $f_t(x) := (1 - t)f(x) + tx_0$. Note that

$$\|x - f(x) - (x - f_t(x))\| = t\|f(x) - x_0\| = t\|f(x) - f(x_0)\| \leq t\|x - x_0\|.$$

It follows that there exists $\delta > 0$ such that for all $x \in \partial H$ and all t with $0 \leq t \leq \delta$,

$$\|x - f(x) - (x - f_t(x))\| \leq \delta\|x - x_0\| < c.$$

This implies that $x - f_t(x) \neq 0$ for all $x \in \partial H$ and $0 \leq t \leq \delta$. The homotopy property (D3) now implies that

$$\text{deg}(\text{id} - f, H, 0) = \text{deg}(\text{id} - f_\delta, H, 0).$$

Because f is nonexpansive,

$$\|f_\delta(x) - f_\delta(y)\| \leq (1 - \delta)\|x - y\|$$

for all $x, y \in \text{cl} H$, which implies that f_δ has at most one fixed point in H . However, by our construction, we have $f_\delta(x_0) = x_0$. If we select a number $r > 0$ such that $B_r(x_0) \subseteq H$, the additivity property (D2) implies that

$$\text{deg}(\text{id} - f_\delta, H, 0) = \text{deg}(\text{id} - f_\delta, \text{int} B_r(x_0), 0).$$

For $x \in \partial B_r(x_0)$ and $\delta \leq t \leq 1$, we have that

$$\begin{aligned} \|x - f_t(x)\| &= \|x - x_0 - (f_t(x) - f_t(x_0))\| \\ &\geq r - \|f_t(x) - f_t(x_0)\| \\ &\geq r - (1 - t)\|x - x_0\| = tr > 0, \end{aligned}$$

so the homotopy property (D3) implies that

$$\deg(\text{id} - f_\delta, \text{int } B_r(x_0), 0) = \deg(\text{id} - x_0, \text{int } B_r(x_0), 0).$$

The homotopy property also guarantees that $\deg(\text{id} - x_0 + tx_0, \text{int } B_r(x_0), tx_0)$ is constant for all $t \in [0, 1]$. Combining this with (D1), we find that

$$\deg(\text{id} - x_0, \text{int } B_r(x_0), 0) = \deg(\text{id}, \text{int } B_r(x_0), x_0) = 1.$$

Thus, under our assumptions on H , we have proved that $\deg(\text{id} - f, H, 0) = 1$.

It is clear that G can be selected as in the statement of Proposition 3.2, and as previously noted, $x - f(x) \neq 0$ for $x \in \partial G$. Thus it follows from the above results that $\deg(\text{id} - f, G, 0) = 1$. To complete the proof, we argue by contradiction and assume that $\text{Fix}(f)$ is not connected. It follows that $\text{Fix}(f) = A \cup B$, where A and B are disjoint, nonempty, compact sets. It follows that there exist disjoint, bounded, open sets H_A and H_B with $A \subset H_A$, $B \subset H_B$, $\text{cl } H_A \subset D$ and $\text{cl } H_B \subset D$. Our previous arguments imply that $\deg(\text{id} - f, H_A, 0) = \deg(\text{id} - f, H_B, 0) = 1$. Writing $H = H_A \cup H_B$, we also have that $\deg(\text{id} - f, H, 0) = 1$. However, the additivity property (D2) implies that

$$1 = \deg(\text{id} - f, H_A \cup H_B, 0) = \deg(\text{id} - f, H_A, 0) + \deg(\text{id} - f, H_B, 0) = 2,$$

a contradiction. ■

If the set D in Proposition 3.2 is convex and f is nonexpansive on $\text{cl } D$, Bruck [11] has proved that $\text{Fix}(f)$ is a nonexpansive retract of D , which gives connectedness of $\text{Fix}(f)$ as a very special case. A simple proof of this result, valid for the finite-dimensional case, is given in [31, §4]. For more information on nonexpansive retracts, see [24].

Combining Propositions 3.1 and 3.2 gives the following corollary.

COROLLARY 3.3: *Let D be an open subset of a finite-dimensional normed space $(V, \|\cdot\|)$. Suppose that $f: D \rightarrow V$ is nonexpansive and $\text{Fix}(f)$ is compact and nonempty. If G is a bounded open set such that $\text{cl } G \subset D$ and $\text{Fix}(f) \subset G$, then for each $y \in V \setminus \{0\}$ there exists $w \in \partial G$ and $\lambda > 0$ such that $w - f(w) = \lambda y$.*

Note that if f in Corollary 3.3 is defined on the whole of V and $p \in V$, we can take G to be any open ball with center p and sufficiently large radius.

We now state the main result of this section.

THEOREM 3.4: *If $f: V \rightarrow V$ is a nonexpansive map on a finite-dimensional normed space $(V, \|\cdot\|)$, then the following statements are equivalent:*

- (1) $\text{Fix}(f)$ is nonempty and bounded.
- (2) There exists a bounded open subset $G \subset V$ such that $x - f(x) \neq 0$ for all $x \in \partial G$ and $\deg(\text{id} - f, G, 0) = 1$.
- (3) For every $y \in V \setminus \{0\}$, there exists $w \in V$ and $\lambda > 0$ such that $w - f(w) = \lambda y$.
- (4) There exist w_1, \dots, w_m in V such that $\{f(w_i) - w_i : i = 1, \dots, m\}$ illuminates $B_1(0)$.
- (5) There exists no horofunction h of $(V, \|\cdot\|)$ such that $h(f(x)) \leq h(x)$ for all $x \in V$.

Proof. The statement (1) implies (2) by Proposition 3.2. The implication (2) implies (3) follows from Proposition 3.1. Also note that (3) implies (4), as the illumination number of $B_1(0)$ is finite. Proposition 2.2 gives that (5) implies (1). All that remains is to prove that (4) implies (5).

Let h be a horofunction in the horofunction compactification of $(V, \|\cdot\|)$ with base point b . We begin by observing that there exists $z \in V, \|z\| = 1$, such that

$$(3.1) \quad h(x + tz) = h(x) - t$$

for all $x \in V$ and $t \in \mathbb{R}$. Indeed, by Lemma 2.1 there is a sequence (y_k) in V such that

$$h(x) = \lim_{k \rightarrow \infty} \|x - y_k\| - \|b - y_k\|$$

and $\|y_k\| \rightarrow \infty$. By passing to a subsequence if necessary, let $z = \lim_{k \rightarrow \infty} y_k / \|y_k\|$. The equation

$$\begin{aligned} \left\| x + t \frac{y_k - x}{\|y_k - x\|} - y_k \right\| - \|b - y_k\| &= \|x - y_k\| \left(1 - \frac{t}{\|x - y_k\|} \right) - \|b - y_k\| \\ &= \|x - y_k\| - \|b - y_k\| - t \end{aligned}$$

becomes equation (3.1) in the limit as $k \rightarrow \infty$.

Now suppose that $\{f(w_i) - w_i : i = 1, \dots, m\}$ illuminates the unit ball $B_1(0)$ in V . Since $z \in \partial B_1(0)$, there is some $i \in \{1, \dots, m\}$ such that $f(w_i) - w_i$ illuminates z . Therefore $z + \varepsilon(f(w_i) - w_i) \in \text{int } B_1(0)$ for some sufficiently small $\varepsilon > 0$. By translation, $z + w_i + \varepsilon(f(w_i) - w_i) \in \text{int } B_1(w_i)$.

As the pointwise limit of a sequence of convex, Lipschitz 1 functions, h must be convex and Lipschitz 1 itself. Since $z + w_i + \varepsilon(f(w_i) - w_i) \in \text{int } B_1(w_i)$ and

h is Lipschitz 1, it follows that

$$h(z + \varepsilon f(w_i) + (1 - \varepsilon)w_i) > h(w_i) - 1 = h(z + w_i)$$

by equation (3.1). By convexity, we obtain from the preceding inequality that

$$(1 - \varepsilon)h(z + w_i) + \varepsilon h(z + f(w_i)) > h(z + w_i),$$

which immediately gives $h(z + f(w_i)) > h(z + w_i)$. Then by equation (3.1), $h(f(w_i)) > h(w_i)$. ■

Theorem 3.4 has the following interesting consequence concerning the space $N(V, \|\cdot\|)$ consisting of all nonexpansive maps on a finite-dimensional normed space $(V, \|\cdot\|)$. For other recent results on generic properties of nonexpansive maps, we refer the interested reader to [35].

COROLLARY 3.5: *The subset of $N(V, \|\cdot\|)$, consisting of those nonexpansive maps $f: V \rightarrow V$ with $\text{Fix}(f)$ nonempty and bounded, is open and dense in the topology of compact convergence on $N(V, \|\cdot\|)$. Moreover, if $f \in N(V, \|\cdot\|)$ is such that $\text{Fix}(f)$ is unbounded, then for each $\delta > 0$ there exists $g \in N(V, \|\cdot\|)$ such that $\text{Fix}(g)$ is empty and*

$$\sup_{x \in V} \|f(x) - g(x)\| \leq \delta.$$

Proof. If $\text{Fix}(f) \subset \text{int } B_R(0)$ is nonempty, then $\deg(\text{id} - f, \text{int } B_R(0), 0) = 1$ by Proposition 3.2. Select $\varepsilon > 0$ such that $\min\{\|x - f(x)\| : x \in \partial B_R(0)\} \geq \varepsilon$ and consider the neighborhood

$$U := \left\{ h \in N(V, \|\cdot\|) : \sup_{x \in \partial B_R(0)} \|f(x) - h(x)\| < \varepsilon/2 \right\}$$

of f in the topology of compact convergence. Let $g \in U$ and define the homotopy $g_t(x) := tg(x) + (1 - t)f(x)$ for $t \in [0, 1]$ and $x \in B_R(0)$. Then for $x \in \partial B_R(0)$ we have that

$$\|x - g_t(x)\| \geq \|x - f(x)\| - \|f(x) - g_t(x)\| \geq \varepsilon - t\|f(x) - g(x)\| > \varepsilon/2 > 0$$

for all $t \in [0, 1]$. So, by (D3) we get that $\deg(\text{id} - g, \text{int } B_R(0), 0) = 1$, and hence (D1) implies that g has a fixed point in $\text{int } B_R(0)$. As $\text{Fix}(g)$ is connected, see [11, Theorems 2 and 3], and g has no fixed points in $\partial B_R(0)$, we conclude that $\text{Fix}(g) \subseteq B_R(0)$, which shows that

$$\{f \in N(V, \|\cdot\|) : \text{Fix}(f) \text{ nonempty and bounded}\}$$

is open. To show that it is dense let $f \in N(V, \|\cdot\|)$ and recall that the topology of compact convergence has a basis of open sets,

$$U(g, A, \varepsilon) := \{h \in N(V, \|\cdot\|) : \sup_{x \in A} \|h(x) - g(x)\| < \varepsilon\},$$

where $g \in N(V, \|\cdot\|)$, $A \subseteq V$ compact, and $\varepsilon > 0$. Let $f_k(x) := (1 - 1/k)f(x)$ for $k > 1$. Note that f_k is a Lipschitz contraction on V , and hence f_k has a unique fixed point. Moreover, for each neighborhood $U(f, A, \varepsilon) := \{h \in N(V, \|\cdot\|) : \sup_{x \in A} \|h(x) - f(x)\| < \varepsilon\}$ of f we have that $f_k \in U(f, A, \varepsilon)$ for all $k > 1$ sufficiently large. This completes the proof of the first part of the corollary.

Now suppose that $\text{Fix}(f)$ is unbounded. By Theorem 3.4(3), there exists $y \in V$, $\|y\| = 1$, such that $w - f(w) \neq \lambda y$ for all $w \in V$ and $\lambda > 0$. Fix some $\delta > 0$ and define $g: V \rightarrow V$ by $g(x) := f(x) + \delta y$ for all $x \in V$. Then $\|g(x) - f(x)\| \leq \delta$ for all $x \in V$ and $\text{Fix}(g)$ is empty, as otherwise there exists $x^* \in V$ with $x^* - f(x^*) = \delta y$, which would be a contradiction. ■

4. Detecting fixed points by illumination

Theorem 3.4 suggests the following test for detecting fixed points of nonexpansive maps $f: V \rightarrow V$ on finite-dimensional normed spaces. Randomly generate a finite set of points S in V . Subsequently check if $\{f(w) - w : w \in S\}$ illuminates the unit ball B_1 . If so, f must have a fixed point and its fixed point set is bounded. This raises the practical question of how to quickly determine whether a set of vectors illuminates the unit ball for a given norm. As we shall see in this section there are many classes of norms, such as smooth norms and polyhedral norms, for which there are simple computational criteria to check whether $\{f(w) - w : w \in S\}$ illuminates the unit ball B_1 .

We start with the following basic observation.

LEMMA 4.1: *Let $K \subseteq V$ be a compact, convex set with nonempty interior. If $x, y \in \partial K$ are such that the line segment $\{tx + (1 - t)y : 0 \leq t \leq 1\} \subseteq \partial K$, and x is illuminated by v , then v illuminates $tx + (1 - t)y$ for all $0 < t \leq 1$. Moreover, if S illuminates every extreme point of K , then S illuminates K .*

Proof. If v illuminates x , then $x + \lambda v \in \text{int } K$ for all $\lambda > 0$ sufficiently small. By convexity of K we find that $tx + (1 - t)y + t\lambda v = t(x + \lambda v) + (1 - t)y \in \text{int } K$ for

all $0 < t \leq 1$. The second assertion now follows from the fact that each $x \in \partial K$ is a convex combination of extreme points of K ; see [37, Corollary 18.5.1]. ■

For norms with a polyhedral unit ball it is easy to check whether the extreme points of the unit ball are illuminated, which is sufficient by Lemma 4.1. This leads to the following simple criterion in the case of the *supremum norm* $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$ on \mathbb{R}^n .

PROPOSITION 4.2: *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be nonexpansive with respect to the supremum norm. $\text{Fix}(f)$ is nonempty and bounded if and only if for each $J \subseteq \{1, \dots, n\}$ there exists $w \in S$ such that $f(w)_j < w_j$ for all $j \in J$ and $f(w)_j > w_j$ for all $j \notin J$.*

Proof. The extreme points of the unit ball in $(\mathbb{R}^n, \|\cdot\|_\infty)$ are the points z^J where $J \subseteq \{1, \dots, n\}$ and $z_j^J = 1$ if $j \in J$ and $z_j^J = -1$ otherwise. Clearly $z^J + \lambda v \in \text{int } B_1$ if and only if $v_j < 0$ for all $j \in J$ and $v_j > 0$ for all $j \notin J$. Thus, $\{f(w) - w : w \in S\}$ illuminates B_1 if and only if for each $J \subseteq \{1, \dots, n\}$ there exists $w \in S$ such that $f(w)_j < w_j$ for all $j \in J$ and $f(w)_j > w_j$ for all $j \notin J$. Therefore the result follows from Theorem 3.4. ■

The following necessary condition for a set to illuminate the unit ball is also sufficient for smooth norms. The observation is closely related to known results; see [9, Corollary 35.3].

LEMMA 4.3: *Let $(V, \|\cdot\|)$ be a finite-dimensional normed space. If $\{v_1, \dots, v_m\} \subset V$ illuminates the unit ball B_1 , then $0 \in \text{int conv}\{v_1, \dots, v_m\}$. If $\|\cdot\|$ is a smooth norm, the converse also holds. That is, $0 \in \text{int conv}\{v_1, \dots, v_m\}$ implies that $\{v_1, \dots, v_m\}$ illuminates B_1 .*

Proof. Suppose that $0 \notin \text{int conv}\{v_1, \dots, v_m\}$. By the Hahn–Banach theorem, there is a linear functional $\varphi \in V^*$, $\|\varphi\|_* = 1$ such that $\varphi(v_i) \geq 0$ for all $1 \leq i \leq m$. There exists $z \in \partial B_1$ such that $\varphi(z) = 1$. Note that $\varphi(z + \varepsilon v_i) \geq \varphi(z)$ for all $\varepsilon > 0$ and $1 \leq i \leq m$. Therefore B_1 is not illuminated by $\{v_1, \dots, v_m\}$.

Now suppose that $\|\cdot\|$ is a smooth norm and $0 \in \text{int conv}\{v_1, \dots, v_m\}$. Choose $z \in \partial B_1$. By the smoothness of $\|\cdot\|$, there exists a unique $\varphi \in V^*$, $\|\varphi\|_* = 1$, such that $\varphi(z) = 1$. For $\varepsilon > 0$ sufficiently small, $-\varepsilon z \in \text{int conv}\{v_1, \dots, v_m\}$. Then

$$-\varepsilon = \varphi(-\varepsilon z) = \lambda_1 \varphi(v_1) + \dots + \lambda_m \varphi(v_m).$$

This means that $\varphi(v_i) < 0$ for some $i \in \{1, \dots, m\}$. Now consider the line $l = \{z + tv_i : t \in \mathbb{R}\}$. Note that $\|z + tv_i\| \geq \varphi(z + tv_i) > 1$ for all $t < 0$. If $\|z + tv_i\| < 1$ for some $t > 0$, then v_i illuminates z . Suppose that is not the case. Then, by the Hahn–Banach theorem, there is a linear functional $\psi \in V^*$, $\|\psi\|_* = 1$, such that $\psi(z + tv_i) \geq \psi(x)$ for all $t \in \mathbb{R}$ and $x \in B_1$. In particular, $\psi(z) = 1$, which implies that $\psi = \varphi$ by the uniqueness of φ . Then $\psi(z + tv_i) \geq 1$ for all t , and therefore $\psi(v_i) = 0$, a contradiction. ■

REMARK 4.4: It is possible to determine whether $0 \in \text{int conv}\{v_1, \dots, v_m\}$ in polynomial time (in both m and the dimension of V) using linear programming.

Solve the linear program:

$$\begin{aligned} &\textbf{maximize} && \varepsilon \in \mathbb{R} \\ &\textbf{subject to} && \lambda_i - \varepsilon \geq 0 \text{ for all } i \in \{1, \dots, m\}, \\ &\textbf{and} && \sum_{i=1}^m \lambda_i v_i = 0, \sum_{i=1}^m \lambda_i = 1. \end{aligned}$$

Then 0 is in the relative interior of $\text{conv}\{v_1, \dots, v_m\}$ if and only if the maximum ε is positive. Furthermore, when $0 \in \text{conv}\{v_1, \dots, v_m\}$, it follows that 0 is in the affine hull of $\{v_1, \dots, v_m\}$. This implies that the affine hull of $\{v_1, \dots, v_m\}$ is the same as the span of $\{v_1, \dots, v_m\}$ by [37, Theorem 1.1]. In particular, the relative interior of $\text{conv}\{v_1, \dots, v_m\}$ is the same as the interior when $\text{span}\{v_1, \dots, v_m\} = V$. This can be checked quickly using a rank computation.

Combining Lemma 4.3 with Theorem 3.4 immediately gives the following result.

COROLLARY 4.5: *Let $f : V \rightarrow V$ be a nonexpansive map on a finite-dimensional smooth normed space $(V, \|\cdot\|)$. There exist $w_1, \dots, w_m \in V$ such that*

$$0 \in \text{int conv}\{f(w_i) - w_i : i = 1, \dots, m\}$$

if and only if $\text{Fix}(f)$ is nonempty and bounded.

For arbitrary norms the following result can be used to verify the existence of a bounded set of fixed points.

PROPOSITION 4.6: *Let $f : V \rightarrow V$ be a nonexpansive map on a finite dimensional normed space $(V, \|\cdot\|)$. If there is a finite collection of vectors $w_i \in V$ with corresponding $v_i := \frac{w_i - f(w_i)}{\|w_i - f(w_i)\|}$ such that the interior of the balls $B_1(v_i)$ covers $\partial B_1(0)$, then $\text{Fix}(f)$ is nonempty and bounded.*

Proof. If z is an extreme point of the unit ball $B_1(0)$, then $\|z - v_i\| < 1$ for some i . Therefore $-v_i$ illuminates z . Since every extreme point is illuminated by some $-v_i$, Theorem 3.4 and Lemma 4.1 imply that f has a nonempty bounded set of fixed points. ■

It is worth asking whether condition (4) of Theorem 3.4 is optimal in the following sense: Suppose that the illumination number of the unit ball B_1 of $(V, \|\cdot\|)$ is K , and $v_1, \dots, v_m \in V$ with $m < K$. Does there exist a nonexpansive map $f: V \rightarrow V$ and points $w_1, \dots, w_m \in V$ with $v_i = f(w_i) - w_i$ for all i , such that $\text{Fix}(f)$ is empty or unbounded? We have the following partial results for this problem.

PROPOSITION 4.7: *If v_1, \dots, v_m are m points in \mathbb{R}^n and $m < 2^n$, then there exist a supremum-norm nonexpansive map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $w_i \in \mathbb{R}^n$ such that $v_i = f(w_i) - w_i$ for all i , and $\text{Fix}(f)$ is unbounded.*

Proof. Suppose that $m < 2^n$. For each i , let $w_i := -v_i$ and define $f(w_i) := 0$. So $v_i = f(w_i) - w_i$ for all i . Let $u := (1, \dots, 1) \in \mathbb{R}^n$. Since $m < 2^n$, there must be one sign pattern in $\{-1, 1\}^n$ that is not equal to the entry-wise sign pattern of any w_i . (Here take the sign of 0 to be positive.) Without loss of generality, assume this is the all-positive sign pattern. Define $f(cu) := cu$ for all $c \geq 0$. Since $\|w_i - cu\|_\infty \geq c$ and $\|f(w_i) - cu\|_\infty = c$ for all i , it follows that f is nonexpansive under $\|\cdot\|_\infty$ on $\{w_i: i = 1, \dots, m\} \cup \{cu: c \geq 0\}$. By a special case of the Aronszajn–Panitchpakdi theorem [3], the map f extends to a supremum-norm nonexpansive map on all of \mathbb{R}^n . ■

For any compact, convex body K in an n -dimensional space it is known that the illumination number is at least $n + 1$; see [9, Theorem 35.1]. Note that this result also follows easily from Lemma 4.3. So, if we are given v_1, \dots, v_m in any n -dimensional normed space V and $m < n + 1$, we can ask if there exists a nonexpansive map $f: V \rightarrow V$ and points $w_1, \dots, w_m \in V$ with $v_i = f(w_i) - w_i$ for all i , such that $\text{Fix}(f)$ is empty or unbounded. The following result gives a positive answer to this question.

PROPOSITION 4.8: *If v_1, \dots, v_m are points in an n -dimensional normed space $(V, \|\cdot\|)$ and $m < n + 1$, then there exist a nonexpansive map $f: V \rightarrow V$ and points $w_1, \dots, w_m \in V$ such that $v_i = f(w_i) - w_i$ for all i , and $\text{Fix}(f)$ is empty or unbounded.*

Proof. First suppose that v_1, \dots, v_m span V . In that case, they form a basis for V since $m \leq n$. Fix $c > 0$ and $w_i := -v_i$ for all i . Then there exists $\varphi \in V^*$ such that $\varphi(w_i) = c$ for all i . Now let z_0 with $\|z_0\| = 1$ be such that $\varphi(z_0) = \|\varphi\|_*$. So, if we let $z := z_0/\|\varphi\|_*$, then $\varphi(z) = \|\varphi\|_*\|z\| = 1$.

Define $f: V \rightarrow V$ by $f(x) = \varphi(x)z - cz$ for $x \in V$. Then

$$\|f(x) - f(y)\| = |\varphi(x - y)|\|z\| \leq \|x - y\|\|\varphi\|_*\|z\| = \|x - y\|$$

for all $x, y \in V$. Note that $f(w_i) = 0$ for all i , so that $v_i = f(w_i) - w_i$. Also $f(\mu z) = \mu z - cz$ for all $\mu \in \mathbb{R}$, and hence $f^k(0) = -kcz$. Since $\{f^k(0) : k \in \mathbb{N}\}$ is unbounded, it follows that $\text{Fix}(f)$ is empty.

If v_1, \dots, v_m do not span V , then there exists $\psi \in V^*$ such that $\psi(v_i) = 0$ for all i and $\psi \neq 0$. Now let f be defined as before with φ replaced by ψ and $c = 0$. Then f is nonexpansive and $f(\mu z) = \mu z$ for all $\mu \in \mathbb{R}$. ■

5. Applications to nonlinear eigenvalue problems

In this section we discuss applications to certain nonlinear eigenvalue problems on cones. In particular, we will consider maps $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$, where $\mathbb{R}_{>0}^n$ is the interior of the standard positive cone $\mathbb{R}_{\geq 0}^n := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}$, that are order-preserving and homogeneous (of degree 1). Recall that $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ is *order-preserving* if $f(x) \leq f(y)$ whenever $x \leq y$. Here \leq is the partial ordering induced by $\mathbb{R}_{\geq 0}^n$, so $x \leq y$ if $y - x \in \mathbb{R}_{\geq 0}^n$. The map f is said to be *homogeneous* if $f(\alpha x) = \alpha f(x)$ for all $\alpha > 0$ and $x \in \mathbb{R}_{>0}^n$.

Particular motivation for studying these maps comes from game theory [1, 7, 38] and mathematical biology [29, 39]. In these applications it is often important to know if f has an eigenvector $x \in \mathbb{R}_{>0}^n$, so $f(x) = \lambda x$ for some $\lambda > 0$. This is equivalent to asking whether the *normalized map* $g_f: \Sigma_0 \rightarrow \Sigma_0$, given by

$$(5.1) \quad g_f(x) := \frac{f(x)}{f(x)_n} \quad \text{for all } x \in \Sigma_0 := \{x \in \mathbb{R}_{>0}^n : x_n = 1\},$$

has a fixed point in Σ_0 .

It is well known [25, Lemma 2.1.6] that f is nonexpansive under *Hilbert's metric*, which is given by

$$d_H(x, y) := \log \left(\max_i \frac{x_i}{y_i} \right) - \log \left(\min_j \frac{x_j}{y_j} \right) \quad \text{for } x, y \in \mathbb{R}_{>0}^n.$$

In fact, Hilbert's metric defines a metric between pairs of rays in $\mathbb{R}_{>0}^n$, as $d_H(\alpha x, \beta y) = d_H(x, y)$ for all $\alpha, \beta > 0$ and $x, y \in \mathbb{R}_{>0}^n$, and $d_H(x, y) = 0$ if

and only if $x = \alpha y$ for some $\alpha > 0$; see [25, Proposition 2.1.1]. Thus, d_H is a metric on Σ_0 and g_f is nonexpansive on Σ_0 .

If $x, y \in \mathbb{R}_{>0}^n$ are eigenvectors with eigenvalues, say ρ and μ , then $\rho = \mu$; see [25, Corollary 5.2.2]. It turns out that our results can be used to analyze the *eigenspace*

$$E(f) := \{x \in \mathbb{R}_{>0}^n : x \text{ is an eigenvector of } f\}.$$

Indeed, we have that $E(f)$ is nonempty and bounded in $(\mathbb{R}_{>0}^n, d_H)$ if and only if $\text{Fix}(g_f)$ is nonempty and bounded in (Σ_0, d_H) . The reader can verify that the coordinate-wise log function is an isometry from (Σ_0, d_H) onto the $(n - 1)$ -dimensional normed space $(V_0, \|\cdot\|_v)$, where $V_0 := \{x \in \mathbb{R}^n : x_n = 0\}$ and $\|x\|_v := \max_i x_i - \min_j x_j$ is the *variation norm* on V_0 ; see [25, §2.2]. It follows that the map $h: V_0 \rightarrow V_0$, given by

$$h(x) = (\text{Log} \circ g_f \circ \text{Exp})(x),$$

is nonexpansive on $(V_0, \|\cdot\|_v)$, and hence we can apply our results to h . Note that the unit ball of $(V_0, \|\cdot\|_v)$ has $2^n - 2$ extreme points, which are given by

$$(5.2) \quad \{v_+^I : \emptyset \neq I \subseteq \{1, \dots, n - 1\}\} \cup \{v_-^I : \emptyset \neq I \subseteq \{1, \dots, n - 1\}\},$$

where $(v_+^I)_i = 1$ if $i \in I$ and 0 otherwise, and $(v_-^I)_i = -1$ if $i \in I$ and 0 otherwise. See [30, §2] or [23, Proposition 3.2] for details.

We begin by using our results to prove a nonlinear Perron–Frobenius-type theorem. Recall that the classical Perron–Frobenius theorem says that if A is a nonnegative n -by- n matrix and A is irreducible, then A has a unique normalized eigenvector $v \in \mathbb{R}_{>0}^n$ with eigenvalue the spectral radius $r(A)$ of A . The following result can be seen as a nonlinear Perron–Frobenius-type theorem and should be compared to [1, Theorem 6], [12], [14, Theorem 2], and [25, Theorem 6.2.3].

THEOREM 5.1: *If $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ is an order-preserving homogeneous map, then $E(f)$ is nonempty and bounded in $(\mathbb{R}_{>0}^n, d_H)$ if and only if for each nonempty proper subset J of $\{1, \dots, n\}$ there exists $x^J \in \mathbb{R}_{>0}^n$ such that*

$$(5.3) \quad \max_{j \in J} \frac{f(x^J)_j}{x_j^J} < \min_{j \in J^c} \frac{f(x^J)_j}{x_j^J}.$$

Proof. Note that $E(f)$ is nonempty and bounded in $(\mathbb{R}_{>0}^n, d_H)$ if and only if $\text{Fix}(g_f)$ is nonempty and bounded, which is equivalent to saying that the nonexpansive map $h = \text{Log} \circ g_f \circ \text{Exp}$ on $(V_0, \|\cdot\|_v)$ has a nonempty and bounded

fixed point set. So, if $E(f)$ is nonempty and bounded, then it follows from Theorem 3.4(3) that for each proper nonempty subset J of $\{1, \dots, n\}$ there exists $y^J \in V_0$ such that

$$(5.4) \quad \max_{j \in J} h(y^J)_j - y_j^J < \min_{j \in J^c} h(y^J)_j - y_j^J.$$

Now let $x^J := \text{Exp}(y^J) \in \Sigma_0$. Then (5.4) is equivalent to

$$\max_{j \in J} (h(\text{Log}(x^J))_j - (\text{Log}(x^J))_j) < \min_{j \in J^c} (h(\text{Log}(x^J))_j - (\text{Log}(x^J))_j),$$

which holds if and only if

$$(5.5) \quad \max_{j \in J} \log g_f(x^J)_j - \log x_j^J < \min_{j \in J^c} \log g_f(x^J)_j - \log x_j^J,$$

where g_f is given in (5.1). Now note that (5.5) holds if and only if

$$\max_{j \in J} \frac{g_f(x^J)_j}{x_j^J} < \min_{j \in J^c} \frac{g_f(x^J)_j}{x_j^J},$$

which is equivalent to (5.3).

Now suppose that (5.3) holds. For each nonempty proper subset J of $\{1, \dots, n\}$ let $y^J \in V_0$ be given by $y^J := \text{Log}(x^J/x_n^J)$. So the inequality (5.4) holds for each y^J .

Note that if v_+^I is an extreme point of B_1 given by (5.2), then for each $\varepsilon > 0$ sufficiently small we have that

$$\|v_+^I + \varepsilon(h(y^I) - y^I)\|_v = 1 + \varepsilon \max_{j \in I} (h(y^J)_j - y_j^J) - \varepsilon \min_{j \in I^c} (h(y^J)_j - y_j^J) < 1,$$

and hence v_+^I is illuminated by $h(y^I) - y^I$. Likewise, for v_-^I we can take $J := \{1, \dots, n\} \setminus I$, so that for all $\varepsilon > 0$ sufficiently small,

$$\|v_-^I + \varepsilon(h(y^J) - y^J)\|_v = \varepsilon \max_{j \in J} (h(y^J)_j - y_j^J) - (-1 + \varepsilon \min_{j \in J^c} (h(y^J)_j - y_j^J)) < 1,$$

which shows that v_-^I is illuminated by $h(y^J) - y^J$. It now follows from Lemma 4.1 and Theorem 3.4 that $\text{Fix}(h)$ is nonempty and bounded, which implies that $\text{Fix}(g_f)$ is nonempty and bounded. ■

REMARK 5.2: Using a case by case analysis it is not hard to show that the illumination number of the unit ball B_1 in the $(n - 1)$ -dimensional normed space $(V_0, \|\cdot\|_v)$ is 3 for $n = 3$, and 6 for $n = 4$. For general n , the first author and Lewis White have recently shown that the illumination number is $\binom{n}{\lfloor n/2 \rfloor}$.

REMARK 5.3: If an order-preserving homogeneous map on $\mathbb{R}_{>0}^n$ is a linear map associated to a nonnegative matrix A , then the eigenspace $E(A)$ is nonempty and bounded in Hilbert's metric if and only if A has a unique (up to scaling) eigenvector in $\mathbb{R}_{>0}^n$.

For a nonnegative n -by- n matrix $A = (a_{ij})$, let $G(A)$ denote the adjacency digraph of A , that is, the graph on vertices $\{1, \dots, n\}$ with an edge from i to j if and only if $a_{ij} > 0$. For $i, j \in \{1, \dots, n\}$, we say that i has *access* to j if there is a path from i to j in $G(A)$. We say that i and j *communicate* if they both have access to each other. Communication is an equivalence relation, and the equivalence classes of $\{1, \dots, n\}$ under communication are called the *classes* of A . A class α is *final* if no vertex $i \in \alpha$ has access to any vertex outside α . It is *basic* if the square submatrix of A corresponding to α has spectral radius equal to the spectral radius of A . A nonnegative matrix A has a positive eigenvector if and only if the final classes of A are exactly its basic classes. Furthermore, the positive eigenvector is unique (up to scaling) if and only if A has only one basic, final class. See [6, Ch. 2, Theorem 3.10 and its proof] for details. Note that the classes and their corresponding spectral radii can be determined with prescribed accuracy in polynomial time as the dimension n grows. This is much faster than verifying the conditions of Theorem 5.1 as in that case the number of constraints to fulfill grows exponentially for linear maps with large n .

We should note that a nonlinear order-preserving homogeneous map $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ can have an eigenspace space $E(f)$ which is bounded in Hilbert's metric and consists of more than a single ray. Simple examples of such maps can be constructed as follows.

Example 5.4: For $0 \leq c \leq \frac{1}{3}$, let

$$\begin{aligned}\mu_{1,c}(x) &:= \max\{x_2, x_3, c(x_1 + x_2 + x_3)\}, \\ \mu_{2,c}(x) &:= \max\{x_1, x_3, c(x_1 + x_2 + x_3)\}, \\ \mu_{3,c}(x) &:= \max\{x_1, x_2, c(x_1 + x_2 + x_3)\}.\end{aligned}$$

Let $f_c: \mathbb{R}_{>0}^3 \rightarrow \mathbb{R}_{>0}^3$ be defined by

$$f_c(x) := \begin{cases} (x_1, \mu_{1,c}(x), \mu_{1,c}(x)) & \text{if } x_1 = \max\{x_1, x_2, x_3\}, \\ (\mu_{2,c}(x), x_2, \mu_{2,c}(x)) & \text{if } x_2 = \max\{x_1, x_2, x_3\}, \\ (\mu_{3,c}(x), \mu_{3,c}(x), x_3) & \text{if } x_3 = \max\{x_1, x_2, x_3\}. \end{cases}$$

It is not hard to verify that f_c is well-defined, order-preserving, and homogeneous. In the case $c = 0$, f_c was described in [29, p.131]. Let $\Sigma := \{x \in \mathbb{R}_{>0}^3 : x_1 + x_2 + x_3 = 1\}$ and let $g_c(x) := f_c(x)/(x_1 + x_2 + x_3)$ for all $x \in \Sigma$. For convenience, let e_1, e_2, e_3 denote the elementary basis vectors in \mathbb{R}^n , so that Σ is the relative interior of $\text{conv}\{e_1, e_2, e_3\}$. For each c , $\text{Fix}(g_c)$ is the union of three line segments, $[\frac{1}{3}(e_1 + e_2 + e_3), (1 - 3c)e_i + c(e_1 + e_2 + e_3)] \cap \Sigma$, $i \in \{1, 2, 3\}$ as shown in Figure 1. If $c = \frac{1}{3}$, then $\text{Fix}(g_c)$ is a single point and therefore f_c has a unique eigenvector in $\mathbb{R}_{>0}^3$ (up to scaling). When $c < \frac{1}{3}$, however, the fixed points of g_c are not unique, nor are the fixed point sets convex. If $c > 0$, $\text{Fix}(g_c)$ is nonempty and bounded in Hilbert’s metric on Σ . If $c = 0$, the fixed point set is unbounded and Theorem 5.1 does not apply.

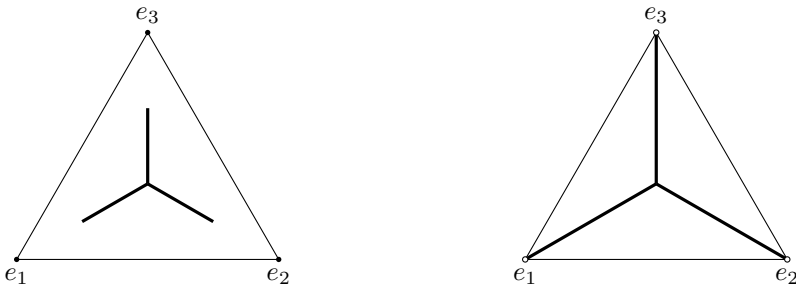


Figure 1. $\text{Fix}(g_c)$ from Example 5.4 when $c = \frac{1}{6}$ (left) and $c = 0$ (right).

Theorem 5.1 suggests the following algorithm for detecting the presence of eigenvectors in $\mathbb{R}_{>0}^n$. Let $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ be an order-preserving homogeneous map.

- Step 1:** Randomly select $x \in \Sigma_0$ and compute $f(x)_j/x_j$ for all $j \in \{1, \dots, n\}$.
- Step 2:** Record all nonempty proper subsets $J \subset \{1, \dots, n\}$ such that inequality (5.3) holds.
- Step 3:** Repeat steps 1 and 2 until every nonempty proper subset J has been recorded.

If the algorithm above halts, then f has an eigenvector in $\mathbb{R}_{>0}^n$. Of course, if the eigenspace of f is empty or unbounded under Hilbert’s metric, then the algorithm will never halt. Note that the second step in the algorithm can

be quickly completed by listing the values of $f(x_j)/x_j$ in ascending order. In particular, each $x \in \mathbb{R}_{>0}^n$ can satisfy inequality (5.3) for at most $n - 1$ different proper non-empty subsets J of $\{1, \dots, n\}$ so the algorithm will need to try at least $\frac{2^n - 2}{n - 1}$ different elements of Σ_0 , which makes it impractical when n is large. However, in low-dimensional spaces the algorithm could be useful.

In the remainder of this section, we describe a class of order-preserving homogeneous maps $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ for which no general methods exist to determine the existence of an eigenvector in $\mathbb{R}_{>0}^n$. For these maps, the algorithm above may be particularly useful. For $r \in \mathbb{R}$ with $r \neq 0$, and $\sigma \in \mathbb{R}_{\geq 0}^n$ with $\sum_i \sigma_i = 1$, define the (r, σ) -mean of $x \in \mathbb{R}_{>0}^n$ by

$$M_{r\sigma}(x) := \left(\sum_{i=1}^n \sigma_i x_i^{1/r} \right)^{1/r},$$

and $M_{0\sigma}(x) := \prod_{i \in \text{supp}(\sigma)} x_i$, where $\text{supp}(\sigma) = \{i: \sigma_i > 0\}$. Furthermore define $M_{\infty, \sigma}(x) := \max\{x_i: i \in \text{supp}(\sigma)\}$ and $M_{-\infty, \sigma}(x) := \min\{x_i: i \in \text{supp}(\sigma)\}$. We say that an order-preserving homogeneous map $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ belongs to M if each coordinate function is of the form

$$f_i(x) = \sum_{(r, \sigma) \in \Gamma_i} c_{i r \sigma} M_{r\sigma}(x),$$

where Γ_i is a nonempty set of pairs (r, σ) , with $r \in [-\infty, \infty]$ and $\sigma \in \mathbb{R}_{\geq 0}^n$ such that $\sum_i \sigma_i = 1$, and each $c_{i r \sigma} > 0$. A map $f \in M$ is said to belong to M_+ if each $r \in [0, \infty)$, and it belongs to M_- if each $r \in (-\infty, 0)$. By closing the sets M , M_+ and M_- under multiplication with positive scalars, addition, and composition, we obtain classes of maps \mathcal{M} , \mathcal{M}_+ , and \mathcal{M}_- , respectively. For maps in \mathcal{M}_+ there exist a variety of general results to determine the existence of an eigenvector in $\mathbb{R}_{>0}^n$. However, no general methods for detecting eigenvectors in $\mathbb{R}_{>0}^n$ are known for maps in \mathcal{M}_- . An extensive discussion of this problem can be found in [25, §6.6] and [29].

In [39], the following map $f: \mathbb{R}_{>0}^4 \rightarrow \mathbb{R}_{>0}^4$ was studied as part of a population biology model:

$$(5.6) \quad f(x) := \begin{pmatrix} a_1 x_1 + b_1 \theta(x_1, x_2) + c_1 \theta(x_1, x_4) + d_1 \theta(x_2, x_3) \\ a_2 x_2 + b_2 \theta(x_1, x_2) + c_2 \theta(x_1, x_4) + d_2 \theta(x_2, x_3) \\ a_3 x_3 + b_3 \theta(x_3, x_4) + c_3 \theta(x_1, x_4) + d_3 \theta(x_2, x_3) \\ a_4 x_4 + b_4 \theta(x_3, x_4) + c_4 \theta(x_1, x_4) + d_4 \theta(x_2, x_3) \end{pmatrix}$$

where the coefficients $b_i, c_i, d_i \geq 0$ for $1 \leq i \leq 4$, with at least one positive for each i , and $\theta(s, t) := (s^{-1} + t^{-1})^{-1}$. In the original model, the coefficients a_i were negative as they represent a ‘force of mortality’. By adding a multiple of the identity to f , we may assume that each a_i , $1 \leq i \leq 4$, is positive without changing the eigenvectors of f . With this additional assumption, $f \in M_-$. In [29, §3], detailed conditions on the coefficients of f are given that determine whether or not f has an eigenvector with positive entries.

Unlike the conditions in [29], the algorithm discussed in this section does not classify all coefficients for which the map above has an entry-wise positive eigenvector. For any particular choice of coefficients, however, the algorithm gives an elementary method for determining whether the map with those coefficients has a positive eigenvector. The advantage of the new algorithm lies in its ability to work with any order-preserving homogeneous map $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$, even ones for which other techniques fail. See [25, §6.2, §6.3, and §6.6] for an overview of other techniques for verifying the existence of entry-wise positive eigenvectors.

Example 5.5: Let f and g be defined by (5.6), where the coefficients of f and g are given by

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 3 & 5 \\ 4 & 3 & 1 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 & 5 & 7 & 2 \\ 3 & 3 & 1 & 1 \\ 4 & 4 & 13 & 1 \\ 1 & 2 & 7 & 8 \end{pmatrix},$$

respectively. The composition $f \circ g \in \mathcal{M}_-$. Known results cannot confirm whether or not $f \circ g$ has an eigenvector in $\mathbb{R}_{>0}^4$. However, our algorithm quickly verifies that $f \circ g$ has eigenvectors in $\mathbb{R}_{>0}^4$. Note that $f \circ g$ is Fréchet differentiable and the derivative $D(f \circ g)(x)$ is a matrix with all positive entries for every $x \in \mathbb{R}_{>0}^4$. This implies that if $f \circ g$ has an eigenvector in $\mathbb{R}_{>0}^4$, then it must be unique up to scaling. See, for example, [25, Corollary 6.4.8]. It also follows from [29, Theorem 3.7] that for any strictly positive vector x in \mathbb{R}^4 , the normalized iterates of $f \circ g$ applied to x converge to the unique normalized eigenvector of $f \circ g$ once one knows that $f \circ g$ has a strictly positive eigenvector. This observation can be exploited to make our algorithm somewhat more efficient for many functions in \mathcal{M} , though we have not attempted to do so here.

To generate test vectors for the algorithm, we randomly selected vectors $w = \text{Exp}(y)$, where $y \in V_0 := \{x \in \mathbb{R}^n : x_n = 0\}$, are vectors that are uniformly distributed in the set $\{y \in V_0 : -R \leq y_j \leq R \text{ for all } 1 \leq j \leq n - 1\}$. For

this example, the value $R = 100$ worked well, but of course, R must be chosen large enough to accommodate the set of eigenvectors of the map. We ran 500 independent trials and recorded the number of test vectors w needed to confirm the existence of a bounded set of eigenvectors. The largest number of test vectors needed was 303, and the smallest was 10. The average was 54.4, and the median was 39. By iterating $f \circ g$ on the vector $(1, 1, 1, 1) \in \mathbb{R}^4$, we find that the unique eigenvector is approximately $(0.24138896, 0.10237913, 0.56235034, 1)$ when normalized so that the last entry is 1.

6. Localizing the fixed point set

Once the presence of a nonempty and bounded set of fixed points has been confirmed, a natural follow up problem is to give bounds on the location of the fixed points. Here we show how this can be accomplished.

If C is a bounded set in a finite-dimensional normed linear space V , define $R_0 \geq 0$ by

$$R_0 := \inf\{R > 0: \bigcap_{x \in C} B_R(x) \neq \emptyset\}$$

and call R_0 the *circumradius* of C . Note that as for each $R > R_0$, $\bigcap_{x \in C} B_R(x)$ is compact, convex and nonempty, therefore $\bigcap_{x \in C} B_{R_0}(x)$ is compact, convex and nonempty. A point $p \in \bigcap_{x \in C} B_{R_0}(x)$ is called a *circumcenter* of C .

Suppose that $(V, \|\cdot\|)$ is a finite-dimensional normed space with unit ball B_1 and suppose that $\{v_1, \dots, v_m\}$ illuminate ∂B_1 . For each $j = 1, \dots, m$, the set $U_j := \{x \in \partial B_1: v_j \text{ illuminates } x\}$ is relatively open in ∂B_1 . Let $U_j^c := \partial B_1 \setminus U_j$ and note that $\max_j d(z, U_j^c) > 0$ for all $z \in \partial B_1$. As ∂B_1 is compact, we find that

$$(6.1) \quad \delta := \min_{z \in \partial B_1} \max_{j=1, \dots, m} d(z, U_j^c) > 0.$$

We also define the following two constants:

$$(6.2) \quad \alpha := \sup_{z \in \partial B_1} d(z, \text{ext } B_1),$$

$$(6.3) \quad \beta := \inf\{\|z - u\|: z \in \partial B_1, u \in \text{ext } B_1 \text{ and } \|\frac{1}{2}z + \frac{1}{2}u\| < 1\}.$$

Here $\text{ext } B_1$ denotes the extreme points of B_1 .

THEOREM 6.1: *Suppose $f: V \rightarrow V$ is a nonexpansive map on a finite-dimensional normed space $(V, \|\cdot\|)$ with unit ball B_1 and there exist $w_1, \dots, w_m \in V$*

such that $\{f(w_i) - w_i : i = 1, \dots, m\}$ illuminates B_1 . If R_0 is the circumradius of $\{w_1, \dots, w_m\}$, p is a circumcenter and δ is defined as in (6.1) where each U_j corresponds to $v_j := f(w_j) - w_j$, then $\text{Fix}(f) \subset B_R(p)$ where $R := (\frac{2+\delta}{\delta})R_0$. Furthermore, if α, β are defined as in (6.2) and (6.3) above and $\beta > \alpha$, then $\delta \geq \beta - \alpha$ and $\text{Fix}(f) \subset B_{R'}(p)$ where $R' := (\frac{2+\beta-\alpha}{\beta-\alpha})R_0$.

Proof. Let R_0 be the circumradius of $\{w_i : i = 1, \dots, m\}$ and let p be a circumcenter for $\{w_i : i = 1, \dots, m\}$, so $\|w_i - p\| \leq R_0$ for $1 \leq i \leq m$. Define $\hat{f}(x) := f(x + p) - p$ and $\hat{w}_i := w_i - p$ for $i = 1, \dots, m$, so \hat{f} is nonexpansive, $\hat{f}(\hat{w}_i) - \hat{w}_i = f(w_i) - w_i$ and $\|\hat{w}_i\| \leq R_0$ for $1 \leq i \leq m$ and $\text{Fix}(\hat{f}) = \{x - p : x \in \text{Fix}(f)\}$. Thus, by replacing f with \hat{f} and w_i by \hat{w}_i , $1 \leq i \leq m$, we may as well assume that $p = 0$ and $\|w_i\| \leq R_0$ for $1 \leq i \leq m$.

Suppose now that $z \in V$ and $\|z\| > R := (\frac{2+\delta}{\delta})R_0$. By (6.1), there is an $i \in \{1, \dots, m\}$ such that $d(z/\|z\|, U_i^c) \geq \delta$.

We claim that

$$\left\| \frac{z - w_i}{\|z - w_i\|} - \frac{z}{\|z\|} \right\| < \delta.$$

To see this, observe that

$$\begin{aligned} \left\| \frac{z - w_i}{\|z - w_i\|} - \frac{z}{\|z\|} \right\| &\leq \frac{\|w_i\|}{\|z - w_i\|} + \|z\| \left\| \frac{1}{\|z - w_i\|} - \frac{1}{\|z\|} \right\| \\ &\leq \frac{R_0}{\|z\| - R_0} + \frac{\| \|z\| - \|z - w_i\| \|}{\|z - w_i\|} \\ &\leq \frac{R_0}{\|z\| - R_0} + \frac{R_0}{\|z\| - R_0} \\ &< \frac{2R_0}{(\frac{2+\delta}{\delta})R_0 - R_0} = \delta. \end{aligned}$$

It follows that $\frac{z - w_i}{\|z - w_i\|} \in U_i$ is therefore illuminated by $f(w_i) - w_i$. Then

$$\frac{z - w_i}{\|z - w_i\|} + \lambda(f(w_i) - w_i) \in \text{int } B_1$$

for some small $\lambda > 0$. Note that the function $\mu : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\mu : t \mapsto \left\| \frac{z - w_i}{\|z - w_i\|} + t(f(w_i) - w_i) \right\|$$

is convex. Since $\mu(0) = 1$ and $\mu(\lambda) < 1$, it follows that $\mu(t) > 1$ for all $t < 0$. In particular, when $t = -1/\|z - w_i\|$ we see that

$$\left\| \frac{z - w_i}{\|z - w_i\|} - \frac{f(w_i) - w_i}{\|z - w_i\|} \right\| > 1$$

so that

$$\|z - f(w_i)\| > \|z - w_i\|.$$

If $f(z) = z$, the inequality above contradicts the nonexpansiveness of f . Thus $\text{Fix}(f) \subset B_R(0)$. Note that the inclusion is strict as $B_R(0)$ contains $\{w_1, \dots, w_m\}$, and none of the w_i are fixed points of f .

Now suppose that α, β are defined as in (6.2) and (6.3), and $\beta > \alpha$. Choose any $z \in \partial B_1$. We will show that $\delta \geq \beta - \alpha$ by showing that

$$(6.4) \quad \max_{1 \leq i \leq m} d(z, U_i^c) \geq \beta - \alpha.$$

First note that there exists an extreme point $u \in B_1$ such that $\|z - u\| \leq \alpha'$ for any α' satisfying $\alpha < \alpha' < \beta$. There is also an $i \in \{1, \dots, m\}$ such that $f(w_i) - w_i$ illuminates u . Now consider any $y \in U_i^c$. Note that $v_i := f(w_i) - w_i$ does not illuminate y , therefore for all $\varepsilon > 0$, $\|y + \varepsilon v_i\| \geq 1$. Then for all sufficiently small $\varepsilon > 0$,

$$\frac{1}{2} \frac{y + \varepsilon v_i}{\|y + \varepsilon v_i\|} + \frac{1}{2} u = \frac{1}{2} \frac{y}{\|y + \varepsilon v_i\|} + \frac{1}{2} \left(u + \frac{\varepsilon v_i}{\|y + \varepsilon v_i\|} \right) \in \text{int } B_1.$$

By (6.3), it follows that

$$\left\| \frac{y + \varepsilon v_i}{\|y + \varepsilon v_i\|} - u \right\| \geq \beta.$$

By taking the limit as $\varepsilon \rightarrow 0$, we see that $\|y - u\| \geq \beta$ as well. By the triangle inequality,

$$\|z - y\| \geq \|y - u\| - \|z - u\| \geq \beta - \alpha'.$$

Taking the minimum over y , then letting α' approach α , we complete the proof of (6.4). Moreover,

$$R' = \left(\frac{2 + \beta - \alpha}{\beta - \alpha} \right) R_0 \geq \left(\frac{2 + \delta}{\delta} \right) R_0 = R,$$

so $\text{Fix}(f) \subset B_{R'}(0)$. ■

REMARK 6.2: For general norms, it is not possible to place uniform lower bounds on δ from (6.1) without additional assumptions on the illuminating set. However, for some polyhedral norms the constants α, β from (6.2) and (6.3) satisfy $\beta - \alpha > 0$, and it is possible to give uniform bounds on $\text{Fix}(f)$ based only on the circumcenter and circumradius of a set $\{w_1, \dots, w_m\} \subset V$ such that $\{f(w_i) - w_i : 1 \leq i \leq m\}$ illuminates the unit ball.

For the supremum norm $\|\cdot\|_\infty$ on \mathbb{R}^n , it is not hard to verify that the constants in (6.2) and (6.3) are $\alpha = 1$ and $\beta = 2$. So, if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is nonexpansive with respect to $\|\cdot\|_\infty$, $\{f(w_i) - w_i : 1 \leq i \leq m\}$ is a set that illuminates the unit ball in $(\mathbb{R}^n, \|\cdot\|_\infty)$, R_0 is the circumradius of $\{w_i : 1 \leq i \leq m\}$, and p is a circumcenter of $\{w_i : 1 \leq i \leq m\}$, then $\text{Fix}(f) \subset B_{3R_0}(p)$.

For the l_1 norm on \mathbb{R}^n , $\alpha = 2 - 2/n$ and $\beta = 2$. We leave the details as an exercise for the reader.

For inner-product spaces the following can be shown.

PROPOSITION 6.3: *If $f : V \rightarrow V$ is a nonexpansive map on an inner-product space V , then*

$$\text{Fix}(f) \subseteq \bigcap_{w \in V} H_w,$$

where $H_w := \{v \in V : \langle v, w - f(w) \rangle \leq \langle w, w - f(w) \rangle\}$. Moreover, if V is finite-dimensional and $\{f(w_i) - w_i : i = 1, \dots, m\}$ illuminates the unit ball of V , then $\text{Fix}(f) \subseteq \bigcap_{i=1}^m H_{w_i}$ and $\bigcap_{i=1}^m H_{w_i}$ is a compact set.

Proof. Suppose that there exist $x \in \text{Fix}(f)$ and $w \in V$ such that $x \notin H_w$. Since f is nonexpansive, $\|x - f(w)\| = \|f(x) - f(w)\| \leq \|x - w\|$, so that

$$\begin{aligned} 0 &\geq \|x - f(w)\|^2 - \|x - w\|^2 \\ &= 2\langle x, w - f(w) \rangle + \|f(w)\|^2 - \|w\|^2 \\ &> 2\langle w, w - f(w) \rangle + \|f(w)\|^2 - \|w\|^2 \\ &= \|w - f(w)\|^2, \end{aligned}$$

which is impossible.

To prove the second part, note that it follows from Lemma 4.3 that $\text{conv}\{f(w_i) - w_i : i = 1, \dots, m\}$ is a compact polytope with 0 in its interior. This implies that $P := \text{conv}\{w_i - f(w_i) : i = 1, \dots, m\}$ is a compact polytope with 0 in its interior. The polar of P is given by

$$P^\circ := \{v \in V : \langle v, w_i - f(w_i) \rangle \leq 1 \text{ for all } i = 1, \dots, m\},$$

which is also a compact polytope with 0 in interior. Now let

$$Q := \{v \in V : \langle v, w_i - f(w_i) \rangle \leq C \text{ for all } i = 1, \dots, m\},$$

where $C := \max\{1, \max_{i=1, \dots, m} \langle w_i, w_i - f(w_i) \rangle\}$, and note that Q is a compact polytope with 0 in interior. Clearly

$$\bigcap_{i=1}^m \{v \in V : \langle v, w_i - f(w_i) \rangle \leq \langle w_i, w_i - f(w_i) \rangle\} \subseteq Q,$$

which completes the proof. ■

By applying Theorem 6.1 to variation norm nonexpansive maps we derive the following result.

THEOREM 6.4: *Suppose $f: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$ is an order-preserving homogeneous map and for each nonempty proper subset J of $\{1, \dots, n\}$ there is an $x^J \in \mathbb{R}_{>0}^n$ satisfying (5.3). If R_0 is the Hilbert metric circumradius of the set $S := \{x^J : J \subset \{1, \dots, n\}, J \neq \emptyset\}$, and p is a circumcenter of S , then $E(f) \subset B_{(2n-1)R_0}(p)$.*

To prove this theorem we need the following lemma.

LEMMA 6.5: *Let B_1 be the unit ball in the $(n - 1)$ -dimensional normed space $(V_0, \|\cdot\|_v)$.*

(a) *For each $w \in \partial B_1$ there exists an extreme point v of B_1 with*

$$\|v - w\|_v \leq 1 - \frac{1}{n-1}.$$

(b) *If $v \in \text{ext } B_1$ and $w \in \partial B_1$ with $\|v - w\|_v < 1$, then $\frac{1}{2}v + \frac{1}{2}w \in \partial B_1$.*

Proof. Let $\mathbb{R}e = \text{span}\{e\}$, where $e = (1, \dots, 1)$. On the quotient space $\mathbb{R}^n/\mathbb{R}e$ we also have the variation norm $\|[x]\|_v = \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i$ for $[x] \in \mathbb{R}^n/\mathbb{R}e$. It is easy to verify that $L: \mathbb{R}^n/\mathbb{R}e \rightarrow V_0$ given by $L[x] = x - x_n e$ is a well-defined linear isometry from $(\mathbb{R}^n/\mathbb{R}e, \|\cdot\|_v)$ onto $(V_0, \|\cdot\|_v)$. Thus, it suffices to show the assertions for the unit ball B_1 in $(\mathbb{R}^n/\mathbb{R}e, \|\cdot\|_v)$.

To prove part (a) let $[w] \in \mathbb{R}^n/\mathbb{R}e$ with $\|[w]\|_v = 1$. We assume without loss of generality that $1 = w_1 \geq w_2 \geq \dots \geq w_n = 0$, as we can relabel the coordinates. Thus, there exists $i < n$ such that

$$w_i - w_{i+1} \geq \frac{1}{n-1}.$$

The extreme points of B_1 in $(\mathbb{R}^n/\mathbb{R}e, \|\cdot\|_v)$ are given by $\{[v] : v \in \{0, 1\}^n \setminus \{e\}\}$. Now let $[v]$ be the extreme point with $v_k = 1$ for all $k \leq i$ and $v_i = 0$ otherwise.

Then

$$\|[v] - [w]\|_v = \|v - w\|_v = 1 - w_i - (-w_{i+1}) \leq 1 - \frac{1}{n-1},$$

which proves part (a).

To prove the second assertion let $[v]$ be an extreme point of B_1 . By relabeling the coordinates, we may assume that $v_k = 1$ for all $k \leq i$ and $v_k = 0$ otherwise. Now suppose that $[w] \in \partial B_1$ and $\|[v] - [w]\|_v < 1$. We can choose a representative for w such that $0 \leq w_k \leq 1$ for all k . As $\|[v] - [w]\|_v < 1$, it follows that $0 < w_1, \dots, w_i \leq 1$ and $0 \leq w_{i+1}, \dots, w_n < 1$. By relabeling the coordinates of w we may furthermore assume that $0 < w_i \leq \dots \leq w_1 \leq 1$ and $0 \leq w_{i+1} \leq \dots \leq w_n < 1$. Recall that $\|[w]\|_v = 1$, so that $w_1 = 1$ and $w_{i+1} = 0$. Then $\frac{1}{2}(w_1 + v_1) = 1$ and $\frac{1}{2}(w_{i+1} + v_{i+1}) = 0$, so $\|\frac{1}{2}[w] + \frac{1}{2}[v]\|_v \geq 1$. ■

Proof of Theorem 6.4. Let Σ_0, g_f, V_0, h be as described in the proof of Theorem 5.1. As observed in the proof of Theorem 5.1, the set

$$\{h(y^J) - y^J : y^J = \text{Log}(x^J/x_n^J) \text{ where } x^J \in S\}$$

illuminates the unit ball in $(V_0, \|\cdot\|_v)$. Since the map Log is an isometry from (Σ_0, d_H) onto $(V_0, \|\cdot\|_v)$, it suffices to prove that $\text{Fix}(h) \subset B_{(2n-1)R_0}(\text{Log}(p))$ in $(V_0, \|\cdot\|_v)$.

By Lemma 6.5, the constants in (6.2) and (6.3) for $(V_0, \|\cdot\|_v)$ are $\alpha \leq 1 - \frac{1}{n-1}$ and $\beta \geq 1$. In fact, we can see that $\alpha = 1 - \frac{1}{n-1}$ and $\beta = 1$ by considering the vector $(1, 1 - \frac{1}{n-1}, 1 - \frac{2}{n-1}, \dots, 0)$. Theorem 6.1 implies that $\text{Fix}(h) \subset B_{(2n-1)R_0}(\text{Log}(p))$ in $(V_0, \|\cdot\|_v)$. Therefore $E(f) \subset B_{(2n-1)R_0}(p)$ in $(\mathbb{R}_{>0}^n, d_H)$. ■

References

- [1] M. Akian, S. Gaubert and A. Hochart, *Hypergraph conditions for the solvability of the ergodic equation for zero-sum games*, IEEE Conference on Decision and Control (Osaka, Japan) (December 15-18, 2015, to appear).
- [2] H. Amann and S. A. Weiss, *On the uniqueness of the topological degree*, Math. Z. **130** (1973), 39–54.
- [3] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439.
- [4] A. F. Beardon, *Iteration of contractions and analytic maps*, J. London Math. Soc. (2) **41** (1990), 141–150.
- [5] A. F. Beardon, *The dynamics of contractions*, Ergodic Theory Dynam. Systems **17** (1997), 1257–1266.

- [6] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, 1994.
- [7] T. Bewley and E. Kohlberg, *The asymptotic theory of stochastic games*, Math. Oper. Res. **1** (1976), 197–208.
- [8] V. G. Boltjanski, *The problem of the illumination of the boundary of a convex body (in Russian)*, Izv. Mold. Fil. AN SSSR **76** (1960), 78–84.
- [9] V. Boltyanski, H. Martini and P. S. Soltan, *Excursions into combinatorial geometry*, Universitext, Springer-Verlag, Berlin, 1997.
- [10] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer-Verlag, Berlin, 1999.
- [11] R. E. Bruck, Jr., *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.
- [12] R. Cavazos-Cadena, *Equivalence of communication and projective boundedness properties for monotone and homogeneous functions*, Nonlinear Anal. **75** (2012), 775–785.
- [13] K. Deimling, *Nonlinear functional analysis*, Dover books on mathematics, Dover Publications, 2013.
- [14] S. Gaubert and J. Gunawardena, *The Perron-Frobenius theorem for homogeneous, monotone functions*, Trans. Amer. Math. Soc. **356** (2004), 4931–4950.
- [15] S. Gaubert and G. Vigeral, *A maximin characterisation of the escape rate of nonexpansive mappings in metrically convex spaces*, Math. Proc. Cambridge Philos. Soc. **152** (2012), 341–363.
- [16] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, Vol. 28, Cambridge University Press, Cambridge, 1990.
- [17] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 83, Marcel Dekker, Inc., New York, 1984.
- [18] I. Gohberg and A. S. Markus, *One problem on covering convex figures by similar figures (in Russian)*, Izv. Mold. Fil. AN SSSR **76** (1960), 87–90.
- [19] M. Gromov, *Hyperbolic manifolds, groups and actions*, in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., Vol. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 183–213.
- [20] H. Hadwiger, *Unge löste Probleme*, Elem. Math. **12** (1957), 121.
- [21] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.
- [22] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71.
- [23] A. Karlsson, V. Metz and G. A. Noskov, *Horoballs in simplices and Minkowski spaces*, Int. J. Math. Math. Sci. (2006), Art. ID 23656, 20.
- [24] E. Kopecká and S. Reich, *Nonexpansive retracts in Banach spaces*, Banach Center Publ., Vol. 77, Polish Acad. Sci. Inst. Math., Warsaw, 2007.
- [25] B. Lemmens and R. Nussbaum, *Nonlinear Perron-Frobenius theory*, Cambridge Tracts in Mathematics, Vol. 189, Cambridge University Press, Cambridge, 2012.

- [26] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
- [27] J. R. Munkres, *Topology*, Featured Titles for Topology Series, Prentice Hall, Incorporated, 2000.
- [28] R. D. Nussbaum, *Hilbert’s projective metric and iterated nonlinear maps*, Mem. Amer. Math. Soc. **75** (1988), iv+137.
- [29] R. D. Nussbaum, *Iterated nonlinear maps and Hilbert’s projective metric. II*, Mem. Amer. Math. Soc. **79** (1989), iv+118.
- [30] R. D. Nussbaum, *Finsler structures for the part metric and Hilbert’s projective metric and applications to ordinary differential equations*, Differential Integral Equations **7** (1994), 1649–1707.
- [31] R. D. Nussbaum, *Periodic points of nonexpansive maps*, in *Idempotency (Bristol, 1994)*, Publ. Newton Inst., Vol. 11, Cambridge Univ. Press, Cambridge, 1998, pp. 231–241.
- [32] A. Papadopoulos, *Metric spaces, convexity and nonpositive curvature*, IRMA Lectures in Mathematics and Theoretical Physics, Vol. 6, European Mathematical Society (EMS), Zürich, 2005.
- [33] A. Papadopoulos and M. Troyanov (eds.), *Handbook of Hilbert geometry*, IRMA Lectures in Mathematics and Theoretical Physics, Vol. 22, European Mathematical Society (EMS), Zürich, 2014.
- [34] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.
- [35] S. Reich and A. J. Zaslavski, *Two porosity theorems for nonexpansive mappings in hyperbolic spaces*, J. Math. Anal. Appl. **433** (2016), 1220–1229.
- [36] M. A. Rieffel, *Group C^* -algebras as compact quantum metric spaces*, Doc. Math. **7** (2002), 605–651 (electronic).
- [37] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970.
- [38] D. Rosenberg and S. Sorin, *An operator approach to zero-sum repeated games*, Israel J. Math. **121** (2001), 221–246.
- [39] R. Schoen, *The two-sex multiethnic stable population model*, Theoret. Population Biol. **29** (1986), 343–364.
- [40] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space.*, Proc. Amer. Math. Soc. **14** (1963), 438–443.
- [41] C. Walsh, *The horofunction boundary of finite-dimensional normed spaces*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 497–507.
- [42] C. Walsh, *The horofunction boundary of the Hilbert geometry*, Adv. Geom. **8** (2008), 503–529.